Homework1

PS 2010

1 Tangent Lines And Rates Of Change

a). To compute the slope, we need to use function $m_{PQ} = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{y_2-y_1}{x_2-x_1}$. Therefore, when $x = -3$, $f(x) = 3$, so we have a point (-3, 3).

- When $x = -3.1$, $f(x) = 3.63$, so we have the second point (-3.1, 3.63). Therefore, $m_{PQ} = \frac{3.63-3}{-3.1-(-3)} = -6.3$.
- When $x = -3.01$, $f(x) = 3.0603$, so we have the second point (-3.01, 3.0603). Therefore, $m_{PQ} = -6.03$.
- When $x = -3.0001$, $f(x) = 3.00060003$, so we have the second point (-3.0001, 3.00060003). Therefore, $m_{PQ} = -6.0003$.
- When $x = -2.9$, $f(x) = 2.43$, so we have the second point (-2.9, 2.43). Therefore, $m_{PQ} = -5.7$.
- When $x = -2.999$, $f(x) = 2.994003$, so we have the second point (-2.999, 2.994003). Therefore, $m_{PQ} = -5.997$.
- When $x = -2.9999$, $f(x) = 2.99940003$, so we have the second point (-2.9999, 2.99940003). Therefore, $m_{PQ} = -5.9997$.

b) From the above calculation, we know when approaches 3 from two side, the value approximately equal to -6. This means the instantaneous rate of change or to say the slope at point $x = -3$ is -6.

Therefore, we could use our slope function again to get the tangent line:

$$-6 = \frac{f(x)-3}{x-(-3)}$$
$$f(x) = -6x - 15$$

2 Limit

$$\lim_{x\to 2} 3x^2 + 5x - 9$$

When $x = 1.9$, $f(1.9) = 11.33$, when $x = 1.99$, $f(1.99) = 12.8303$. Meanwhile, when $x = 2.1$, $f(2.1) = 14.73$, when $x = 2.01$, $f(2.01) = 13.1703$. Because from both side, the function values approach 13 when x close to 2, the limit should be 13.

$$\lim_{x\to 1} \frac{x^2+2x-3}{x-1}$$

When $x = 0.9$, $f(0.9) = 3.9$, when $x = 0.99$, $f(0.99) = 3.99$. Meanwhile, when $x = 1.1$, $f(1.1) = 4.1$, when $x = 1.01$, $f(1.01) = 4.01$. Because from both side, the function values approach 4 when x close to 1, the limit should be 4.

$$\lim_{x\to a} \frac{x^3-a^3}{x-a}$$

First of all, $x^3 - a^3$ can written as $(x-a)(x^2+ax+a^2)$. Therefore, the function can rewritten as

$$\lim_{x\to a} \frac{(x-a)(x^2+ax+a^2)}{x-a} = x^2+ax+a^2$$

When $x = 0.9a$, $f(0.9a) = 2.71a^2$, when $x = 0.99a$, $f(0.99a) = 2.9701a^2$. Meanwhile, when $x = 1.1a$, $f(1.1a) = 3.31a^2$, when $x = 1.01a$, $f(1.01a) = 3.0301a^2$. Because from both side, the function values approach $3a^2$ when x close to a, the limit should be $3a^2$.

$$\lim_{x\to 1} \begin{cases} x-5 & x \neq 1 \\ 7 & x = 1 \end{cases}$$

When $x = 0.9$, $f(0.9) = -4.1$, when $x = 0.99$, $f(0.99) = -4.01$. Meanwhile, when $x = 1.1$, $f(1.1) = -3.9$, when $x = 1.01$, $f(1.01) = -3.99$. Because from both side, the function values approach -4 when x close to 1, the limit should be -4.

3 Numerical Derivative

(a)

$$f(x) = x^2$$

According the numerical derivative function, we have:

$$\begin{aligned} f'(x) &= \frac{(x+h)^2 - x^2}{h} \\ f'(x) &= \frac{x^2 + 2xh + h^2 - x^2}{h} \\ f'(x) &= \frac{2xh + h^2}{h} \\ f'(x) &= 2x + h \end{aligned}$$

Then, we know when h approaches 0, $f'(x) = 2x + h$ equal to $f'(x) = 2x$

(b)

$$f(x) = 10 + 5x - x^2$$

According the numerical derivative function, we have:

$$\begin{aligned} f'(x) &= \frac{10 + 5(x+h) - (x+h)^2 - (10 + 5x - x^2)}{h} \\ f'(x) &= \frac{10 + 5x + 5h - x^2 - 2hx - h^2 - 10 - 5x + x^2}{h} \\ f'(x) &= \frac{5h - 2hx - h^2}{h} \\ f'(x) &= 5 - 2x + h \end{aligned}$$

Then, we know when h approaches 0, $f'(x) = 5 - 2x + h$ equal to $f'(x) = 5 - 2x$

4 Differentiation

(a)

$$f(x) = 6x^3 - 9x + 4$$
$$f^{'}(x) = 6 * 3x^{3-1} - 9x^{1-1}$$
$$f^{'}(x) = 18x^2 - 9$$

(b)

$$f(x) = \frac{4x^3 - 7x + 8}{x}$$
$$f^{'}(x) = \frac{(4x^3 - 7x + 8)^{'} * x - x^{1-1} * (4x^3 - 7x + 8)}{x^2}$$
$$f^{'}(x) = \frac{(12x^2 - 7) * x - (4x^3 - 7x + 8)}{x^2}$$
$$f^{'}(x) = \frac{8x^3 - 8}{x^2}$$

(c)

From the product rule, we know

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$$

Therefore, $\frac{d}{dx}[f(2)g(2)] = 17 * 3 - (-4)(-8) = 83$

(d)

$$f(x) = (4x^2 - x)(x^3 - 8x^2 + 12)$$
$$f^{'}(x) = (4x^2 - x)^{'}(x^3 - 8x^2 + 12) + (4x^2 - x)(x^3 - 8x^2 + 12)^{'}$$
$$f^{'}(x) = (4 * 2x^1 - 1)(x^3 - 8x^2 + 12) + (4x^2 - x)(3x^2 - 8 * 2x^1)$$
$$f^{'}(x) = (8x - 1)(x^3 - 8x^2 + 12) + (4x^2 - x)(3x^2 - 16x)$$

(e)

$$f(x) = \frac{6x^2}{2 - x}$$
$$f^{'}(x) = \frac{(6x^2)^{'}(2 - x) - 6x^2(2 - x)^{'}}{(2 - x)^2}$$
$$f^{'}(x) = \frac{(12x)(2 - x) + 6x^2 * (-1)}{(2 - x)^2}$$
$$f^{'}(x) = \frac{24x - 6x^2}{(2 - x)^2}$$

(f)

$$f(x) = 2\mathbf{e}^x - 8^x$$
$$f^{'}(x) = 2\mathbf{e}^x - 8^x ln(8)$$

(g)

$$f(x) = 4\log_3(x) - \ln(x)$$
$$f^{'}(x) = 4 * \frac{1}{x(ln(3))} - \frac{1}{x}$$

(h)

$$f(x) = \left(4x^2 - 3x + 2\right)^{-2}$$
$$\text{Let us use}$$
$$f(x) = (g(x))^{-2}$$
$$g(x) = 4x^2 - 3x + 2$$
$$\text{then, we have}$$
$$f^{'}(x) = -2g(x)^{-3} * g^{'}(x)$$
$$g^{'}(x) = 8x - 3$$
$$\text{Finally, we put together and have}$$
$$f^{'}(x) = -2(4x^2 - 3x + 2)^{-3} * (8x - 3)$$

4.1 (i)

$$f(x) = \ln(1 - 5x^2 + x^3)$$
$$\text{Let us use}$$
$$f(x) = \ln(g(x))$$
$$g(x) = 1 - 5x^2 + x^3$$
$$\text{then, we have}$$
$$f^{'}(x) = \frac{1}{g(x)} * g^{'}(x)$$
$$g^{'}(x) = -10x + 3x^2$$
$$\text{Finally, we put together and have}$$
$$f^{'}(x) = \frac{-10x + 3x^2}{1 - 5x^2 + x^3}$$

5 Extrema

Find the local and global extrema for $f(x) = 8x^3 + 81x^2 - 42x - 8$ on $[-8, 2]$

First of all, we need to find the critical points of the function by take the first derivative

$$f^{'}(x) = 24x^2 + 162x - 42 = 6(4x - 1)(x + 7)$$

Second, we let $f^{'}(x) = 0$, then we we have:

$$x = -7 \text{ or } x = \frac{1}{4}$$

Third, we calculate the second derivative

$$f^{''}(x) = 48x + 162$$

Fourth, we can know that $f^{''}(-7) = -174$, and $f^{''}(\frac{1}{4}) = 174$. Because $f^{''}(-7) < 0$, so it is local maximum, also $f^{''}(\frac{1}{4}) > 0$, so it is local minimum.

Fifth, we need to calculate function values at critical points, and at interval boundary when $x = -8$ and $x = 2$

$$f(-8) = 1416,\ f(-7) = 1511.\ f(\frac{1}{4}) = -13.3125,\ f(2) = 296$$

Therefore, the global minimum is $f(\frac{1}{4}) = -13.3125$. the global maximum is $f(-7) = 1511$.