PS 2010: 2. Derivative

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- Recitation time: Friday, 8:30-9:45 at 4801
- Extra Material (click me)
- The first Homework Assignment is out. Due: Sep 8th, 8:59:59am

Agenda

- Motivation
- Tangent Line
- Limit
- Rules of Differentiation
- Partial and Total Derivative
- Optimization

Motivation

- Political scientist interested in marginal effect of...
  - unemployment rate on voter turnout
  - economic performance on government trust
  - ethnic diversity on conflict
  - return to interest groups lobbying for a judicial nomination
- We also want to know
  - decision to join collective actions
  - best defense and offense strategies during conflicts
  - optimal sentencing length to minimize repeat offenses
- Derivatives, differentiation, and optimization provide analytical tools

Tangent Lines

Definition: A tangent line to the function $f(x)$ at the point $x = a$ is a line that just touches the graph of the function at the point in question and is "parallel" (in some way) to the graph at that point.

Two Key Words:

- Just touches the graph
- For a given value on x

Tangent Lines

Figure: This is a tangent line

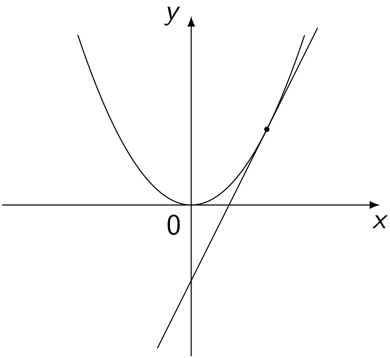
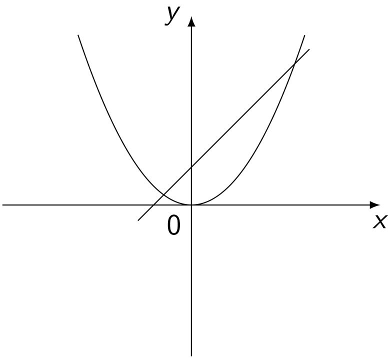


Figure: This is not a tangent line



- However, the line in the right graph is a **secant line.**

Calculate Slope

- We need to know how to calculate the slope of a line:
$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
- Example: $f(x) = 3x$, x1 $= 1$, x2 $= 2$

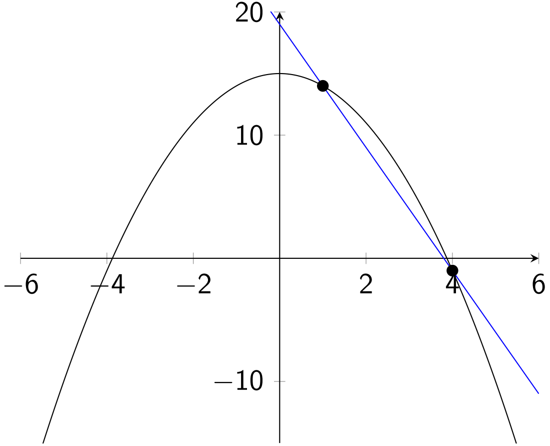
$$m = \frac{f(2) - f(1)}{2 - 1}$$
$$m = \frac{6 - 3}{2 - 1}$$
$$m = 3$$

Find Tangent Line

- $f(x) = 15 - x^2$, at $x = 1$
- Logic: to find a line, we need to find at least two points on the line or we know the slope of the line.
- We already have one point $x = 1 \Rightarrow (1, 14)$
- So the next job is to find the slope, then we can define a line

Step 1

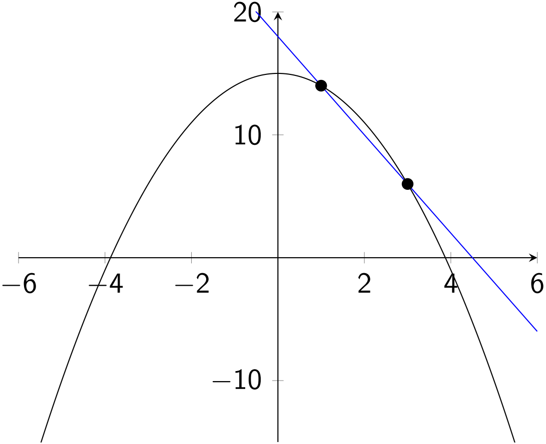
Let use point when x = 4, that is y = -1



It is a secant line, not tangent. So let's decrease to x = 3

Continue Step 1

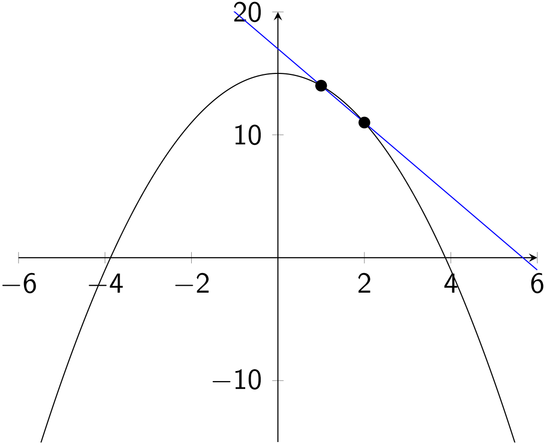
When $x = 3$, that is $y = 6$, we draw the line again



Pretty close but not good enough, let's try $x = 2$

Continue Step 1

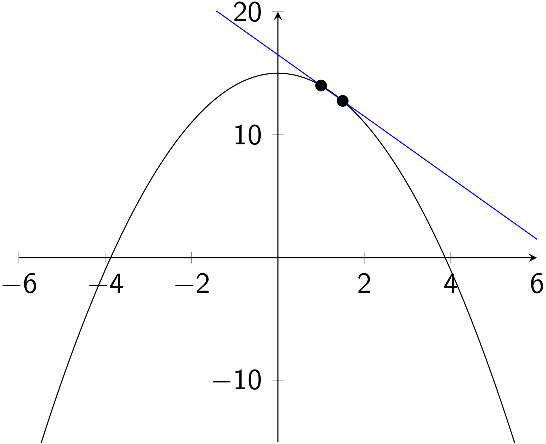
When x = 2, that is y = 11, we draw the line again



Pretty close but not good enough, let's try x = 1.5

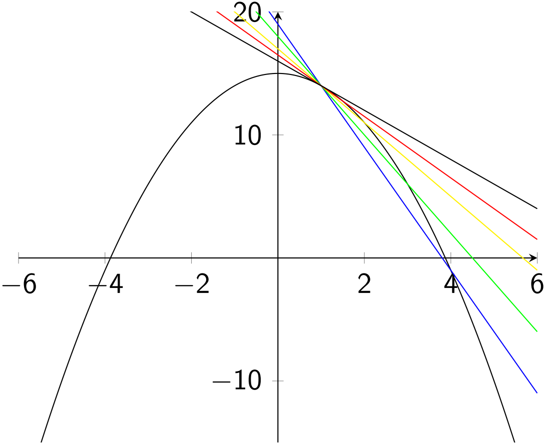
Continue Step 1

When $x = 1.5$, that is $y = 12.75$



Pretty close but not good enough, let's try $x = 1.5$

Pattern: when x approaches to 1, the secant lines also approach to the tangent line.



Final Step

- If so, let x = 1.001, then y = 13.998
- Then, we have two point (1, 14), and (1.001, 13.998)
- We can get slope of the secant line, which is very very very close to the slope of tangent line

$$m = \frac{13.998-14}{1.001-1} = \frac{-0.002}{0.001} = -2$$

- Finally, we have slope = -2 and a point (1, 14), we can define:

$$
\begin{aligned}
-2 &= \frac{f(x) - 14}{x - 1} \\
-2(x-1) &= f(x) - 14 \\
-2x + 2 &= f(x) - 14 \\
f(x) &= -2x + 16
\end{aligned}
$$

Summary

- Another example here
- In all examples, we approach the point from one direction. Results are the same when approach from both left and right.
- Remember: they are tangent lines at given points
- The slope of the tangent reflect the **instantaneous rate of change** at a point.

Limit

- When we choose x value (e.g $x = 1.001$) that approach to certain value (e.g $x = 1$)
- And we plug in this value ($x = 1.001$) to function and get the output
- If we do this from both side, that is $x = 1.001$ and $x = 0.999$, and get function outputs
- Then, this is actually taking the limit of a function at a point.

Definition of Limit

Definition: We say that the limit of f(x) is L as x approaches point h and write this as:

$$\lim_{x\to h} f(x) = L$$

provided we can make f(x) as close to L as we want **for all x sufficiently close to h**, from both sides, **without actually letting x be h**.

- This is not precise definition of a limit, but it is enough for our class.

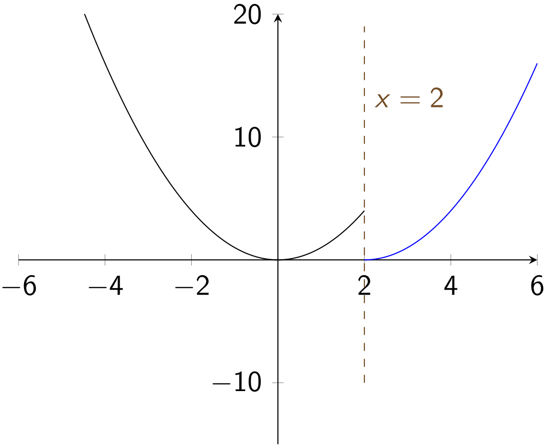
Examples

- $\lim_{x \to 2} x + 4$
- $\lim_{x \to 2} x^2$
- $\lim_{x \to 1} \frac{x^2 - 3x + 5}{x - 2}$

Special Cases

Notice: Limits are asking what the function is doing around x = h, we are not concerned with what the function is actually doing at x = h.

- $\lim_{x \to 2} \frac{8-x^3}{x^2-4}$
- $f(x) = \begin{cases} x^2 & : x < 2 \\ (x-2)^2 & : x \geq 2 \end{cases}$


20
10
−10
−6
−4
−2
2
4
6
x = 2

Properties of Limits

- $\lim_{x \to h}(f(x) + g(x)) = \lim_{x \to h} f(x) + \lim_{x \to h} g(x)$
- $\lim_{x \to h}(f(x) - g(x)) = \lim_{x \to h} f(x) - \lim_{x \to h} g(x)$
- $\lim_{x \to h}(f(x) * g(x)) = \lim_{x \to h} f(x) * \lim_{x \to h} g(x)$
- $\lim_{x \to h} \frac{f(x)}{g(x)} = \frac{\lim_{x \to h} f(x)}{\lim_{x \to h} g(x)}$

Derivative

- **Definition:** The instantaneous rate of change of a function (also means the slope of tangent line) at a given x.

- Formally, The derivative of $f(x)$ with respect to x is the function $f^{'}(x)$ and is defined as,

$$f^{'}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- We also write it as: $\frac{d}{dx} f(x)$ or $\frac{dy}{dx}$

Example

- $f(x) = 3 - 14x^2$
  - Is the function increasing or decreasing when x = 1
  - Is the function increasing or decreasing when x = -1
- Determine $f'(0)$ for $f(x) = |x|$

Differentiation

- A function $f(x)$ is called differentiable at x = a if $f^{'}(a)$ exists
- f(x) is is called differentiable on an interval if the derivative exists for each point in that interval.
- Discontinuity at a point implies no limit at that point, which further implies not differentiable at that point.

Rules of Differentiation 1

- The derivative of a constant is zero: If $f(x) = c$, then $f^{'}(x) = 0$
- **The power rule:** If $f(x) = x^n$, then $f^{'}(x) = nx^{n-1}$
- Examples:
  - $f(x) = 2x^6 + 7x^{-6}$
  - $f(x) = \sqrt{x} + 9\sqrt[3]{x^7} - \frac{2}{\sqrt[5]{x^2}}$

Rules of Differentiation 2

- **Product Rule:** If the two functions $f(x)$ and $g(x)$ are differentiable, then the product is differentiable and,
$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$$
- **Quotient Rule:** If the two functions $f(x)$ and $g(x)$ are differentiable, then the quotient is differentiable and,
$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g(x)^2}$$
- Examples:
  - $f(x) = \left(6x^3 - x\right)(10 - 20x)$
  - $f(x) = \frac{3x+9}{2-x}$

Rules of Differentiation 3

- **Derivatives Of Exponential and Logarithm Functions:**
  - $f(x) = e^x$ then, $f^{'}(x) = e^x$
  - $f(x) = a^x$ then, $f^{'}(x) = a^x ln(a)$
  - $f(x) = ln(x)$ then, $f^{'}(x) = \frac{1}{x}$
  - $f(x) = log_a(x)$ then, $f^{'}(x) = \frac{1}{xln(a)}$
- Examples:
  - $f(x) = 4^x - 5\log_9 x$
  - $f(x) = 3\mathbf{e}^x + 10x^3 \ln x$

Rules of Differentiation 4

- **Chain Rule:** If two functions $f(x)$ and $g(x)$ are both differentiable, then
$$[f(g(x))]' = f'(g(x))g'(x)$$
- Or If we have $y = f(u)$ and $u = g(x)$, then
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$
- Examples:
  - $f(x) = \mathbf{e}^{x^4 - 3x^2 + 9}$
  - $f(x) = \ln\left(x^{-4} + x^4\right)$

Partial Derivative

- A partial derivative is the derivative with respect to one of the variables
- Example: $f(x, y, z) = x^2 + xy + 2yz$

Summary of Rules of Differentiation

Table 6.1: List of Rules of Differentiation

Sum rule	$(f(x)+g(x))' = f'(x)+g'(x)$
Difference rule	$(f(x)-g(x))' = f'(x)-g'(x)$
Multiply by constant rule	$f'(ax) = af'(x)$
Product rule	$(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$
Quotient rule	$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
Chain rule	$(g(f(x))' = g'(f(x))f'(x)$
Inverse function rule	$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$
Constant rule	$(a)' = 0$
Power rule	$(x^n)' = nx^{n-1}$
Exponential rule 1	$(e^x)' = e^x$
Exponential rule 2	$(a^x)' = a^x(\ln(a))$
Logarithm rule 1	$(\ln(x))' = \frac{1}{x}$
Logarithm rule 2	$(\log_a(x))' = \frac{1}{x(\ln(a))}$
Trigonometric rules	$(\sin(x))' = \cos(x)$
	$(\cos(x))' = -\sin(x)$
	$(\tan(x))' = 1+\tan^2(x)$
Piecewise rules	Treat each piece separately

Interpretation of First Derivatives

First derivative, $f^{'}(x)$

- Slope of a tangent line
- Also provide you the instantaneous rate of change
- Implies a function is increasing or decreasing, and by how much it is increasing or decreasing
- A fancier term: marginal effect!

Example

$$f(x) = 3x^3 - 6x^2 + 2x$$

- $f^{'}(x) = 9x^2 - 12x + 2$
- So the instantaneous rate of change when x = 0 is 2
- This also means the function is **increasing** at that point
- So the instantaneous rate of change when x = 1 is -1
- This also means the function is **decreasing** at that point

Critical Point

- **Definition:** We say that $x = c$ is a critical point of the function $f(x)$ if $f(c)$ **exists** and if either of the following are true.

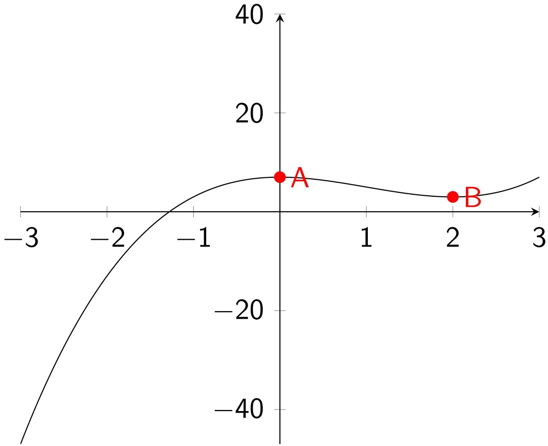
$$f^{'}(c) = 0 \text{ or } f^{'}(c) \text{ doesn't exist}$$

- **Critical points are important because local extrema occur at critical point.**

Example

$$f(x) = x^3 - 3x^2 + 7$$

- $f^{'}(x) = 3x^2 - 6x$
- $f^{'}(x) = 0 \Rightarrow x^* = 0$ or $x^* = 2$
- Let's plot it



- From the plot, we know A and B are local maxima and local minima
- However, we can not plot the function every time.
- We need the second derivative to determine

Second Derivative

Second derivative, $f''(x)$

- The derivative of the derivative of a function
- Change in slope
- Implies first derivative is increasing or decreasing

Second Derivative Test

- After knowing the $f^{'}(x)$, take the derivative again and find $f^{''}(x)$
- Substitute the critical point x^* into $f^{''}(x)$
  - If $f^{''}(x^*) < 0$, $f(x)$ has a local maximum at $x*$
  - If $f^{''}(x^*) > 0$, $f(x)$ has a local minimum at $x*$
- Repeat the above steps for all critical points.

Example

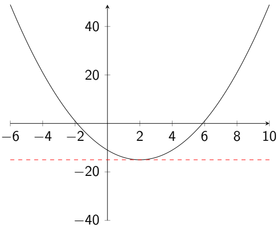
$$f(x) = x^3 - 3x^2 + 7$$

- $f^{'}(x) = 3x^2 - 6x$
- $f^{'}(x) = 0 \Rightarrow x^* = 0$ or $x^* = 2$
- $f^{''}(x) = 6x - 6$
- $f^{''}(0) = -6 \Rightarrow$ local maximum
- $f^{''}(2) = 6 \Rightarrow$ local minimum

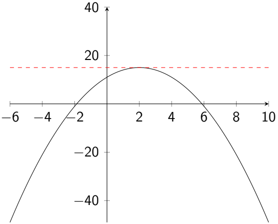
Why Second Derivative

- $f''(x)$ tell us whether function is **concave** or **convex** at certain points
- $f''(x) < 0$ means concave near x
- $f''(x) > 0$ means convex near x
- So, when $f''(x) < 0$ at critical point $\Rightarrow$ local maximum
- When $f''(x) > 0$ at critical point $\Rightarrow$ local minimum

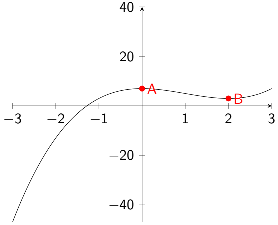
Convex at x = 2



Concave at x = 2

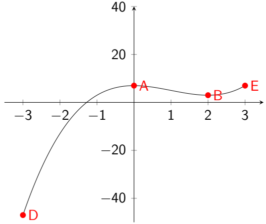


Global Extrema



- The above procedures find the local extrema
- To find global extrema, we also need to look at the boundaries of the domain.

Global Extrema



- The above procedures find the local extrema
- To find global extrema, we also need to look at the boundaries of the domain.
- For example, look at D and E where x=-3 and 3
- Check both critical points and domain boundaries for global extrema

Summary

To find local/global extrema, you need

- Find $f'(x)$
- Set $f'(x) = 0$ and solve for all critical points x^*
- Find $f''(x)$
- For each critical point, substitute it into $f''(x)$
  - If $f''(x^*) < 0$, $f(x)$ has a local maximum at $x*$
  - If $f''(x^*) > 0$, $f(x)$ has a local minimum at $x*$
  - $f''(x^*) = 0$ is also possible, but too complicated for this class.
- Substitute each critical points into $f(x)$ to find the function's value
- Substitute the lower and upper bounds of the domain into f(x) to find the function's values
- The smallest value is the global minimum and the largest value is global maximum.