PS 2010: 3. Integrals

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- Homework 1 Due today
- Homework 2 is out
  Due Sep 19th 8:59:59am
- For next week, read 9 and skim 10

Today's Agenda

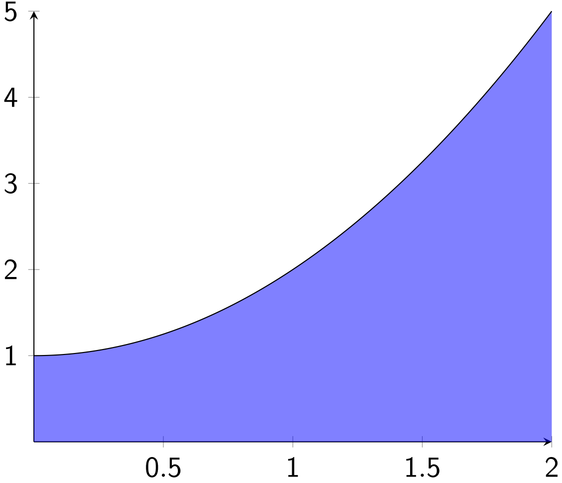
- Why matters
- Riemann Sum
- Indefinite Integral
- Definite Integral

Why We Need to Learn Integrals

- Calculate expected utility in theoretical models, such as voting and conflicts
- Compute probabilities, expected values, and variances associated with distributions
- Calculate likelihood functions and derive maximum likelihood estimators

Area Problem

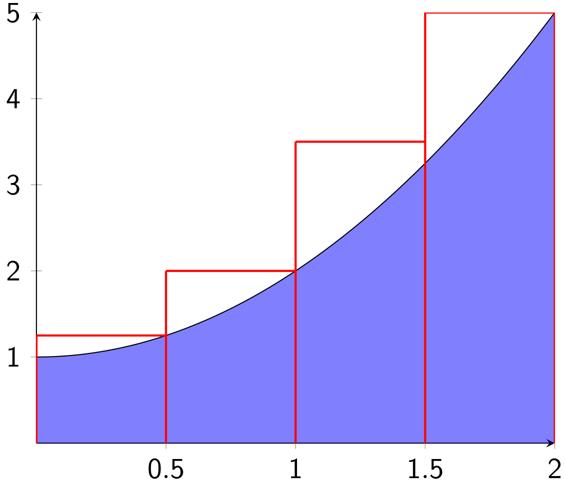
Determine the area of the blue region between the function:
$f(x) = x^2 + 1$ on [0,2] and the x-axis.



Solution

- One way is to divide the blue region into small rectangles.
- First, decide how many rectangles, e.g. 4
- Second, this means x-asix equally divided to four intervals.
- $\Delta x = \frac{\text{Upper bond} - \text{Lower bond}}{n} = \frac{2-0}{4} = 0.5$

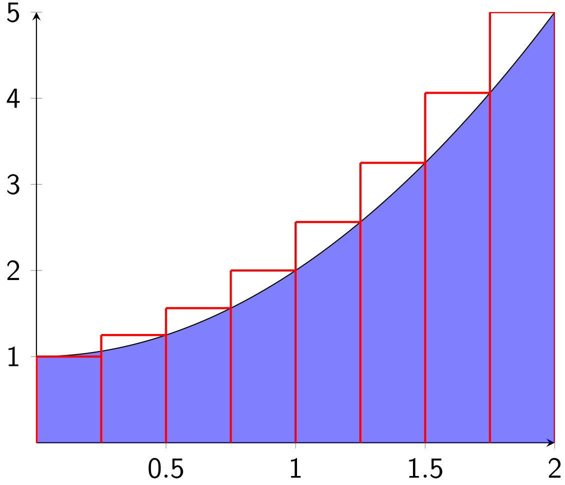
First Try



- Use the function value from right end point as height
- Calculate the each area and sum them up
- $\Delta * f(0.5) + \Delta * f(1) + \Delta * f(1.5) + \Delta * f(2)$

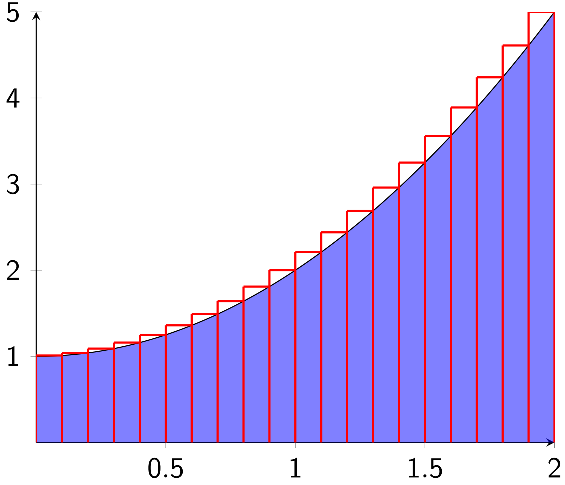
Use Large n

Let's make $n = 8$



A Better Approximation

Let's make $n = 20$



Riemann Sum

- With $\Delta x \to 0$ (n goes to larger and larger)
- The summation of rectangles area approximate to the area under the curve
- **Riemann Sum**:

  $$A = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

  When n goes to infinity we will get the exact area under the curve.
- The above can be also written as:

  $$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

Indefinite Integral

- Suppose we have a function $F(x)$, and it is has derivative $F^{'}(x) = f(x)$
- Then the $F(x)$ is called the anti-derivative of $f(x)$
- The most general anti-derivative of $f(x)$ is called an **indefinite integral** and denoted:

  $\int f(x)\,dx = F(x) + C$, where C is an arbitrary number
- $\int$ called integral symbol, $f(x)$ is integrand

Example 1

$$\int x^4 + 3x - 9dx$$

- The anti-derivative of x^4 is $\frac{1}{5}x^5$
- The anti-derivative of $3x$ is $\frac{3}{2}x^2$
- The anti-derivative of 9 is $9x$
- Therefore, the result is $\frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + C$
- **Don't forget the constant number C**

Rules of Indefinite Integral 1

- $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$
- $\int (f(x) - g(x))dx = \int f(x)dx - \int g(x)dx$
- $\int af(x)dx = a \int f(x)d(x)$

Rules of Indefinite Integral 2

Remember, indefinite integral is just the reverse of the derivative. Therefore, for each derivative rule, we have integral rule.

- **Integral of the constant:** $\int k dx = kx + C$, C and k are all constant
- **Integral of the power:** $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$
- Example:
  - $\int 5x^3 - 10x^{-6} + 4dx$
  - $\int x^8 + x^{-8} dx$

Rules of Indefinite Integral 3

- **Integral of exponential:** $\int \mathbf{e}^x dx = \mathbf{e}^x + C$
- **Integral of logarithm:** $\int a^x dx = \frac{a^x}{\ln a} + C$
- **Integral of logarithm:** $\int \frac{1}{x} dx = \int x^{-1} dx = \ln|x| + C$
- Examples:
  - $\int 4x^3 - 9 + 7e^x$
  - $\int 3^x + \frac{1}{6x}$

Integration by Substitution

If function F and $g(x)$ is differentiable, and we have $F^{'} = f$, then

$$\int f(g(x))g^{'}(x)dx = \int f(u)du = F(g(x)) + C$$

Where $u = g(x)$.

- $\int 3(8x - 1)\mathbf{e}^{4x^2 - x}dx$
- $\int \frac{x}{\sqrt{1-4x^2}}dx$
- $\int \frac{3x}{(5x^2+4)^2}dx$

Integration by Parts

- We know product rule: $(fg)' = f'g + fg'$
- We take integral on both side: $\int (fg)' dx = \int (f'g + fg')dx$
- Then, $fg = \int f'gdx + \int fg'dx$
- Let's further let $f(x) = u$, and $g(x) = v$
- We will have $f'(x) = \frac{du}{dx} \Rightarrow du = f'(x)dx$
- Do the same: $g'(x) = \frac{dv}{dx} \Rightarrow dv = g'(x)dx$
- We get: $uv = \int vdu + \int udv$

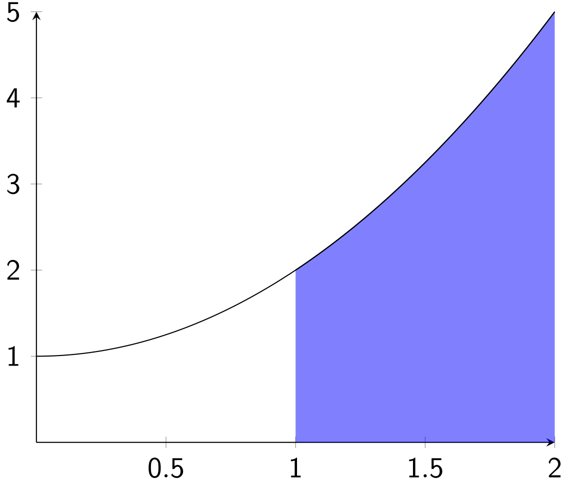
Integration by Parts:

$$\int vdu = uv - \int udv \text{ or } \int udv = uv - \int vdu$$

Examples

- Rule of thumb: order for choosing u: 1) log function, 2) power function, 3) exponential function.
- $\int xe^{6x}dx$
- $\int x \ln xdx$

Area Problem Again



Area of blue region = Area under the curve - Area of the white region.

Definite Integral

- **Definite Integral:** If $f(x)$ is continuous on interval [a, b], we divide the interval to n subintervals of equal width, Δx, and from each interval choose a point x^*, then the integral of $f(x)$ from a and b is:

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

- **It is still an area problem, which sum over all small rectangle areas. The difference is this time we look at area within an interval**

Fundamental Theorem of Calculus

If $f(x)$ is continuous on [a, b], and $F(x)$ is the anti-derivative for $f(x)$. Then

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

- $\int_0^2 x^2 + 1dx$
- $\int_{-3}^1 6x^2 - 5x + 2dx$

More Examples

- Integral by substitution
  - $\int_{-2}^{0} 2x^2\sqrt{1-4x^3}dx$
  - $\int_{-2}^{-6} \frac{4}{(1+2x)^3} - \frac{5}{1+2x}dx$
- Integral by parts
  - $\int_{0}^{1} x^2 e^x dx$

Resource

Integral Calculator