PS 2010: 5. Random Variable 1

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- You all did very well for the assignment 1 and 2
- Middle term exam next week
- All you need to know is in slides and homework assignment
- Extra Online Material
  - Introduction to Probability
  - Foundations of Statistics with R

Agenda

- Expectation and Variance
- Distribution of discrete random variable
- Distribution of continuous random variable

Random Variable

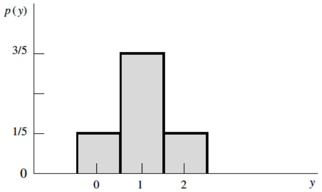
- **Random Variable** assigns a numeric value to each event of the experiment.
- Values need to be mutually exclusive and exhaustive events
- Two types of random variables:
  - Discrete random variable: takes a finite or at most countably infinite number of distinct values
  - Continuous: uncountably infinite number of values.

Probability Distribution for Discrete Random Variable

- Probability that Y takes on the value y is $P(Y = y)$ or $p(y)$
- Probability distribution for a discrete variable provides $p(y) = P(Y = y)$ for all y
- For discrete random variable, it is also **probability mass function, pmf**

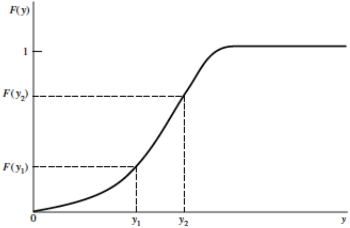
Table 3.1 Probability distribution for Example 3.1

y	$p(y)$
0	1/5
1	3/5
2	1/5



Probability Distribution for Continuous Random Variable

- Probability that Y takes on a specific value y is $P(Y = y) = p(y) = 0$
- Probability distribution for a continuous variable denoted by $F(y)$ is such that $F(y) = P(Y \leq y)$ for all $-\infty < y < \infty$
- The function $p(y)$ is **probability density function, pdf** for continuous variable



Cumulative Distribution Function

- The **cumulative distribution function** $F(y)$ of a discrete random variable y with probability distribution $f(x)$ is
$$F(y) = P(Y \leq y) = \sum_{t \leq y} f(y)$$
- The cumulative distribution function $F(y)$ of a continuous random variable y with probability distribution $f(x)$ is
$$F(y) = P(Y \leq y) = \int_{-\infty}^{y} f(t)dt$$
for all $-\infty < y < \infty$
- $\int_{-\infty}^{\infty} f(x)dx = 1$

$f(x)$ is the probability mass function or probability density function. $f(x) = p(x)$ just different notation.

$$P(a < X < b) = \int_a^b f(x)\, dx.$$

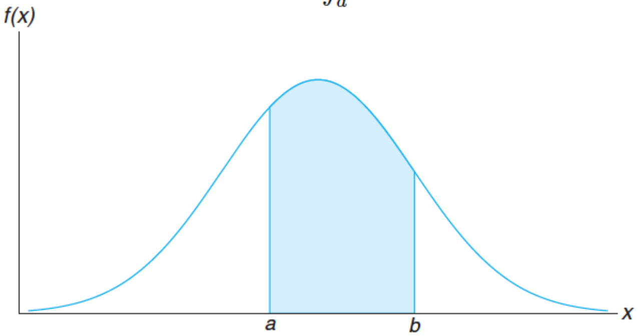


Figure 3.5: $P(a < X < b)$.

Example

In a political survey, it is found that 50% of respondents strongly support a particular policy aimed at enhancing national security

- Find a formula for the probability distribution of the number of respondents who strongly support the policy among the next 4 individuals surveyed.
- Find the cumulative distribution function of the random variable

Example

For a continuous random variable representing political campaign contributions, the probability density function is given by:

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

- Find the probability that a contributor will make campaign contributions over the entire possible range, $P(-\infty < X \leq \infty)$
- Find the probability that a contributor will make campaign contributions between \$0 and \$1, $P(0 < X \leq 1)$

Expectation

- Y is a discrete random variable, then the expected value of Y denoted as $E(Y)$ is $\mu = E(y) = \sum yf(y)$
- If Y is a continuous variable, then $\mu = E(y) = \int_{-\infty}^{\infty} yf(y)$
- Expectation represent the mean of random variables
- Let Y be a random variable with pdf f(y), the expected value of the discrete random variable g(Y) is

$$\mu_{g(Y)} = E[g(Y)] = \sum_y g(y)f(y)$$

- The expected value of the continuous random variable is

$$\mu_{g(Y)} = E[g(Y)] = \int_{-\infty}^{\infty} g(y)f(y)dy$$

Example

In a political poll, a surveyor selects a random sample of 3 individuals from a group of 7 potential voters. Among the group, 4 individuals support a particular candidate, while 3 individuals do not. Calculate the expected number of individuals in this sample who support the candidate.

Example

Let X be the random variable that represents the tenure in months of a particular government official in office. The probability density function is given by:

$$f(x) = \begin{cases} \frac{20{,}0}{x^3}, & x > 100 \\ 0, & \text{elsewhere.} \end{cases}$$

Calculate the expected tenure of this type of government official in months.

Example

Suppose that the number of voters X who arrive at a polling station between 4 : 00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:

x	4	5	6	7	8	9
$P(X = x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the number of voters who express their support for a particular political candidate. Find the expected number of voters who support the candidate during this particular time period.

Example

Let X represent the random variable that measures the level of public trust in a political institution. The density function is given by:

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Calculate the expected change in public trust, denoted by $g(X) = 4X + 3$, due to a certain political event or policy change.

Rules of Expectation

Let X and Y be random variables, and a and b be arbitrary constants. Then

- $E(a) = a$
- $E(aX) = aE(X)$
- $E(aX + b) = aE(X) + b$
- $E(aX + bY) = aE(X) + bE(Y)$
- If X and Y are independent, then $E(XY) = E(X)E(Y)$, but generally, $E(XY) \neq E(X)E(Y)$

Example

A busy student must complete 3 problem sets before doing laundry. Each problem set requires 1 day with probability 2/3 and 2 days with probability 1/3. Let B be the number of days a busy student delays laundry. What is $E(B)$?

Variance

Definition: If X is a random variable with mean $E(X) = \mu$, the variance of a random variable X is defined to be the expected value of $(X - \mu)^2$:

$$V(X) = E[(X - \mu)^2]$$

The square root of X is called the standard deviation

Example: The probability distribution for a random variable Y is given in the table below. Find the mean, variance, and standard deviation of Y.

y	$p(y)$
0	1/8
1	1/4
2	3/8
3	1/4

Variance

$$\begin{aligned} V(X) &= E[(X - E(X))^2] \\ &= E[X^2 - 2X * E(X) + \{E(X)\}^2] \\ &= E(X^2) - 2E(X)E(X) + \{E(X)\}^2 \\ &= E(X^2) - 2\{E(X)\}^2 + \{E(X)\}^2 \\ &= E(X^2) - \{E(X)\}^2 \end{aligned}$$

where we call $E(X)$ as the first moment, and $E(X^2)$ as the second moment

Example: A pdf $f(x) = \frac{1}{b-a}$ at interval [a, b], calculate the variance of random variable X

Rules of Variance

Let X and Y be random variables, and a and b be arbitrary constants. Then

- $V(a) = 0$
- $V(aX) = a^2 V(X)$
- $V(X + b) = V(X)$
- $V(aX + b) = a^2 V(X)$
- If X and Y are independent, $V(X + Y) = V(X) + V(Y)$