Homework4

PS 2010

Due: Oct 3th, 9:29:59am

Notice: The grading will mainly focus on the steps not the final results. This means even your final results are wrong but you follow the correct steps, you will be given the full score.

1 Expectation and Variance

1. x can be 0, 1, 2, 3. We can substitute these numbers to f(x) function to calculate:

$$f(0) = \binom{3}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^3 = 1 * 1 * \frac{27}{64} = \frac{27}{64}$$

$$f(1) = \binom{3}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^2 = 3 * \frac{1}{4} * \frac{9}{16} = \frac{27}{64}$$

$$f(2) = \binom{3}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^1 = 3 * \frac{1}{16} * \frac{3}{4} = \frac{9}{64}$$

$$f(3) = \binom{3}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^0 = 1 * \frac{1}{64} * 1 = \frac{1}{64}$$

Therefore, the mean of X is

$$E(x) = 0 * \tfrac{27}{64} + 1 * \tfrac{27}{64} + 2 * \tfrac{9}{64} + 3 * \tfrac{1}{64} = \tfrac{3}{4}$$

(b)

$$E(x^2) = 0 * \tfrac{27}{64} + 1 * \tfrac{27}{64} + 4 * \tfrac{9}{64} + 9 * \tfrac{1}{64} = \tfrac{8}{9}$$

(c)

$$V(x) = E(x^2) - [E(x)]^2 = \tfrac{8}{9} - \tfrac{9}{16} = \tfrac{9}{16}$$

2. (a) We require a value for c such that

$$F(\infty) = \int_{-\infty}^{\infty} f(y)dy = 1$$
$$= \int_0^2 cy^2 dy = \frac{cy^3}{3}\bigg]_0^2 = \left(\frac{8}{3}\right)c.$$

Thus, $(8/3)c = 1$, and we find that $c = 3/8$.

(b) $P(1 \le Y \le 2) = \int_1^2 f(y)dy = \frac{3}{8}\int_1^2 y^2 dy = \left(\frac{3}{8}\right)\frac{y^3}{3}\Big]_1^2 = \frac{7}{8}$

(c)

$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} y f(y) dy \\ &= \int_0^2 y\left(\frac{3}{8}\right) y^2 dy \\ &= \left(\frac{3}{8}\right)\left(\frac{1}{4}\right) y^4\Big]_0^2 = 1.5. \end{aligned}$$

The variance of Y can be found once we determine $E\left(Y^2\right)$. In this case,

$$\begin{aligned} E\left(Y^2\right) &= \int_{-\infty}^{\infty} y^2 f(y) dy \\ &= \int_0^2 y^2\left(\frac{3}{8}\right) y^2 dy \\ &= \left(\frac{3}{8}\right)\left(\frac{1}{5}\right) y^5\Big]_0^2 = 2.4 \end{aligned}$$

Thus, $\sigma^2 = V(Y) = E\left(Y^2\right) - [E(Y)]^2 = 2.4 - (1.5)^2 = 0.15$.

3. Using the properties of a variance, and independence, we get

$$\operatorname{Var}(Z) = \operatorname{Var}(3X - Y - 5) = \operatorname{Var}(3X - Y) = \operatorname{Var}(3X) + \operatorname{Var}(-Y) = 9\operatorname{Var}(X) + \operatorname{Var}(Y) = 11$$

4. (a) $Cov(Y_1, Y_1) = V(Y_1) = 2$

(b) if $Cov(Y_1, Y_2) = 7$, then, $\rho = \frac{7}{4} > 1$, so it is not possible.

5. we write

$$E\left(X^2 + X - 2\right) = E\left(X^2\right) + E(X) - E(2).$$

Also, $E(2) = 2$, and by direct integration,

$$E(X) = \int_1^2 2x(x-1)dx = \frac{5}{3} \text{ and } E\left(X^2\right) = \int_1^2 2x^2(x-1)dx = \frac{17}{6}.$$

Now

$$E\left(X^2 + X - 2\right) = \frac{17}{6} + \frac{5}{3} - 2 = \frac{5}{2},$$

2 Binomial Distribution

5. Let X be the number of people who survive.

(a) $P(X \ge 10) = 1 - P(X < 10) = 1 - \sum_{k=0}^{9}\binom{n}{k} 0.4^k(1-0.4)^{n-k} = 1 - 0.9662 = 0.0338$

(b)

$$\begin{aligned} P(3 \le X \le 8) &= \sum_{k=3}^{8}\binom{n}{k} 0.4^k(1-0.4)^{n-k} = \sum_{k=0}^{8}\binom{n}{k} 0.4^k(1-0.4)^{n-k} - \sum_{k=0}^{2}\binom{n}{k} 0.4^k(1-0.4)^{n-k} \\ &= 0.9050 - 0.0271 = 0.8779 \end{aligned}$$

(c)

$$P(X = 5) = \binom{15}{5} 0.4^5(1-0.4)^{15-5} = 0.1859$$

(d) $\mu = (15)(0.4) = 6$ and $\sigma^2 = (15)(0.4)(0.6) = 3.6$

3 Negative Binomial

6. (a) Because defective probability is 0.1, so the nondefective probability is 0.9. Therefore, the probability that the first nondefective engine will be found on the second trial is $0.1 * .09 = 0.09$

(b) on the fifth trial is

$$P(x=5) = \begin{pmatrix} 5-1 \\ 2-1 \end{pmatrix} 0.1^2(1-0.1)^{5-2} = 0.02916$$

Meanwhile, the mean is $\mu = E(X) = \frac{K}{p} = \frac{2}{0.1} = 20$, and variance is $\sigma^2 = V(X) = \frac{2(1-0.1)}{0.1^2} = 180$

4 Normal Distribution

7. (a) $P(X > 11.5) = P\left(Z > \frac{11.5-8}{1.5}\right) = P(Z > 2.33) = 1 - 0.9901 = 0.0099$

(b) $P(6.2 < X < 7) = P\left(\frac{6.2-8}{1.5} < Z < \frac{7-8}{1.5}\right) = P(-1.2 < Z < -0.67) = 0.2514 - 0.1151 = 0.1363$

(c) $z = \frac{x-\mu}{\sigma} \Rightarrow 0.845 = \frac{x-8}{1.5} \Rightarrow x = 9.27$

8. (a)$P(x > 500) = P(x > \frac{500-527}{112}) = P(z > -0.24) = 1 - 0.4052 = 0.5948$

(b) This question ask what x makes an individual above 5%, the equivalent question is what score must be to make an individual below 95%. We know that z score is 1.645. Therefore $z = \frac{x-\mu}{\sigma} \Rightarrow 1.645 = \frac{x-527}{112} \Rightarrow x = 711.24$